Statistical Treatment of Transverse Crack Propagation in Aligned Composites

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This paper theoretically examines the failure process of unidirectionally reinforced fiber composites with a transverse notch under a tensile load. The fiber stress distribution around the crack tip and the fiber stress redistribution during the initial failure steps have been obtained based upon the explicit solutions of two-dimensional crack problems by the shear-lag method. Failure sequence analysis is conducted to study the effect of variability in fiber strength on the failure process of the composite. Numerical results show that the failure steps ahead of the first intact fiber are not negligible when the Weibull shape parameter of fiber strength distribution is less than four.

Introduction

THE tensile failure of a unidirectionally reinforced fiber composite is a complex process that involves an accumulation of microstructural damage. In such a material, fibers are relatively weakly coupled by the matrix so that failure of one fiber does not generally precipitate immediate failure of the composite as a whole. Due to this redundancy of the composite's structure, its failure has been considered as a statistical process in which strength is expressed as some measure of the probability of failure such as the mean or median stress.

Zweben and Rosen[1,2] first analyzed the tensile failure of a fiber reinforced composite in terms of crack instability. However, due to the complexity of the problem, they were unable to arrive at a general failure criterion and proposed a conservative approximation of the failure load. Recently, Harlow and Phoenix outlined the complexity of the problem[3] and have obtained exact expressions[4] for the probability of failure of small bundles of fibers under the simplified assumption of stress distribution around fiber fractures. Approximate solutions have also been obtained for larger bundles.[5-11]

In this paper we examine the failure process of unidirectionally reinforced fiber composites with a transverse slit notch, precisely considering the stress field around the crack tips. Based upon the explicit solution[12] of two-dimensional crack problems by the shear-lag method,[13,14] we obtain the fiber stress around the crack and fiber stress redistribution after several fibers ahead of the crack tip have failed. The knowledge of the fiber stress distribution is incorporated into the failure sequence analysis. In the present analysis, we assume the elastic behavior of the matrix material. However, discussion based upon the inelastic analysis[15] will be reported in a future publication. The primary purpose of the study is to provide some insight into the mechanism of failure of notched composites.

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Fiber Stress Concentrations Around a Transverse Notch

General Formulation

The analysis considers a unidirectional continuous fiber composite containing a slit notch in the transverse direction under the uniform load P per fiber at infinity (Fig. 1). The fiber direction is taken along the y axis. The broken fibers are denoted as $n=1,2,3,...,b$, starting from the left tip of the notch with b being the total number of fibers in the notch.

Under the assumption of the shear-lag analysis, the matrix material transfers only shear stress $\bar{\tau}_n(y)$ between two adjacent fibers. Thus $\bar{\tau}_n(y)$ is related to the difference of displacement $u_n^b(y)$ in the fiber direction as

$$\bar{\tau}_n(y) = (Gh/d)\{u_{n+1}^b(y) - u_n^b(y)\} \quad (1)$$

where G is the shear modulus of the matrix, h the thickness of the composite per one fiber, and d the fiber spacing. The tensile force $p_n(y)$ in the nth fiber is related to the displacement by

$$p_n(y) = EA\frac{du_n^b(y)}{dy} \quad (2)$$

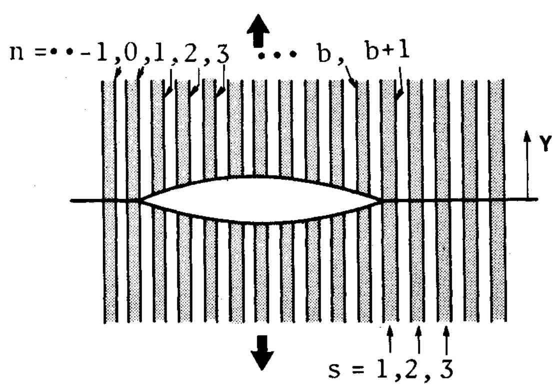

Fig. 1 Model of a multifilament crack in a undirectional composite under uniform force at infinity.

where EA is the extensional stiffness of the fibers. The equilibrium equation of the nth fiber is expressed as

$$EA\frac{d^2u_n^b(y)}{dy^2}+\frac{Gh}{d}[u_{n+1}^b(y)+u_{n-1}^b(y)-2u_n^b(y)]=0 \qquad (3)$$

The transverse displacements of the fibers are not taken into account in the analysis.

The boundary conditions are

$$p_n(0)=0(1\le n\le b) \qquad (4a)$$

$$u_n^b(0)=0(n\le 0,n\ge b+1) \qquad (4b)$$

$$p_n(\pm\infty)=P \quad (\text{all } n) \qquad (4c)$$

Following Hedgepeth,[13,14] the nondimensionalized displacement, axial stress, and coordinates are given by

$$P_n(\xi)=p_n(y)/P$$

$$U_n^b(\xi)=u_n^b(y)(EAGh/dP^2)^{1/2}$$

$$\xi=(Gh/EAd)^{1/2}\cdot y \qquad (5)$$

Thus, Eqs. (3) and (4) become

$$\frac{d^2U_n^b(\xi)}{d\xi^2}=2U_n^b(\xi)-U_{n+1}^b(\xi)-U_{n-1}^b(\xi) \qquad (6)$$

and

$$P_n(0)=0 \qquad (1\le n\le b) \qquad (7a)$$

$$U_n^b(0)=0 \qquad (n\le 0,n\ge b+1) \qquad (7b)$$

$$P_n(\pm\infty)=1 \qquad (\text{all } n) \qquad (7c)$$

Analysis

The differential-difference equation (6) can be reduced to a differential equation by introducing the new function $\bar{V}(\theta,\xi)$, which has the normalized displacement $\{U_n^b(\xi)-\xi\}$ as the Fourier coefficient in its Fourier series expansion

$$\bar{V}(\theta,\xi)=\sum_{n=-\infty}^{+\infty}\{U_n^b(\xi)-\xi\}e^{in\theta} \qquad (8)$$

$$U_n^b(\xi)-\xi=\frac{1}{2\pi}\int_0^{2\pi}\bar{V}(\theta,\xi)e^{-in\theta}d\theta \qquad (9)$$

The equilibrium equation (6) is transformed to

$$\frac{\partial^2\bar{V}(\theta,\xi)}{\partial\xi^2}=\gamma^2\bar{V}(\theta,\xi) \qquad (10)$$

where

$$\gamma=2|\sin(\theta/2)| \qquad (11)$$

Using the boundary conditions [Eqs. (7b) and (7c)], $\bar{V}(\theta,\xi)$ is derived in terms of the unknown constant $U_n^b(0)$, for $\xi\geqq 0$

$$\bar{V}(\theta,\xi)=e^{-\gamma\xi}\cdot\sum_{n=1}^{b}U_n^b(0)e^{in\theta} \qquad (12)$$

The half-crack opening displacements, $U_n^b\equiv U_n^b(0)$, are determined by Eq. (7a). That is,

$$P_n(0)=1-\frac{1}{2\pi}\int_0^{2\pi}\gamma\cdot\sum_{m=1}^{b}U_n^be^{i(m-n)\theta}d\theta=0 \qquad (13)$$

In the matrix formulation, the above equation is expressed as

$$\begin{bmatrix}U_1^b\\U_2^b\\\vdots\\U_b^b\end{bmatrix}=[\tilde{\tilde{G}}]^{-1}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \qquad (14)$$

where the components of the matrix $[\tilde{\tilde{G}}]$ are

$$G_{nm}=\frac{1}{2\pi}\int_0^{2\pi}\gamma e^{i(m-n)\theta}d\theta=\frac{4}{\pi}\frac{1}{\{1-4(m-n)^2\}} \qquad (15)$$

In Hedgepeth's papers[13,14] the influence function L_{nm} is defined as the axial stress in the nth fiber when unit displacement takes place at the mth fiber. G_{nm} is defined as $-L_{nm}$.

Using Eq. (14), the solution of Eq. (6) is

$$U_n^b(\xi)=\xi+\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{-in\theta-\gamma\xi}[e^{i\theta},e^{2i\theta},\ldots,e^{bi\theta}]$$

$$\cdot[\tilde{\tilde{G}}]^{-1}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \qquad (16)$$

The inverse matrix of $[\tilde{\tilde{G}}]$ in Eq. (14) for an arbitrary number of broken fibers is difficult to obtain; however, the general form of U_n^b is obtained explicitly using Legendre polynomials

$$U_n^b=\pi\{2(b-n)+1\}!!\{2n-1\}!!/2^{2b}\{(b-n)!(n-1)!\}^2 \qquad (17)$$

The proof of Eq. (17) is given in a previous paper.[12]

The stress concentration factors of fibers ahead of the crack tip of the crack plane are derived by inserting Eq. (17) into Eq. (16). The stress concentration factor in the $(b+s)$th fiber is denoted as K_b^s. From the general solution of Eq. (16) we obtain[12]

$$K_b^s\equiv P_{b+s}(0)=(b+2s-1)$$

$$\cdot\frac{2s(2s+2)\cdot(2s+4)\ldots(2s+2b-2)}{(2s-1)\cdot(2s+1)\cdot(2s+3)\ldots(2s+2b-3)\cdot(2s+2b-1)} \qquad (18)$$

As a special case of Eq. (18), the stress concentration factor in the first intact fiber ($s=1$) adjacent to b broken fibers is

$$K_b^1=\frac{4\cdot 6\cdot 8\ldots(2b+2)}{3\cdot 5\cdot 7\ldots(2b+1)} \qquad (19)$$

which is the same as Hedgepeth's solution.[14]

Note that the fiber stress concentration factor K_b^s around the crack tip is given as a product of the stress concentration factor in the first intact fiber K_b^1 and a local factor, $(2s-3)!!/(2s-2)!!$, when s is sufficiently small compared to the crack size b,

$$K_b^s=K_b^1\cdot\frac{(2s-3)!!}{(2s-2)!!} \qquad (s=1,2,3\ldots) \qquad (20)$$

The half-crack opening displacement U_n^b is also approximated as

$$U_n^b = U_l^b \cdot \frac{(2n-1)!!}{(2n-2)!!} \tag{21}$$

under the condition $n \ll b$, where

$$U_l^b \cong K_b^l \tag{22}$$

Equations (20-22) indicate that the stress and displacement field around the crack tip is characterized by a single parameter K_b^l. Furthermore, Eq. (20) shows that the stress concentration factor K_b^s decreases in proportion to $s^{-0.5}$ for the small value of s compared to b. These results resemble the findings of fracture mechanics in continuum media for the mode I crack.

Stress Redistribution Due to the Crack Propagation

To solve the successive fiber failure problems, one needs to obtain the fiber stress redistribution during the failure process. The general method of obtaining the fiber stress distribution for an arbitrary set of broken fibers was first formulated by Hedgepeth.[13,14] In his method the unknown variables are the half-crack open displacements of the failed fibers that are to be solved from the linear equations. The result gives the stress distribution with the aid of the influence function. However, this method is rather impracticable when the crack size is large. Furthermore, it is almost impossible to obtain the analytical form that sheds light on the stress redistribution after several fibers ahead of the crack tip have failed. Thus, we converted the above problem in the following way in which the unknown variables are the stress in the unfailed fibers around the crack tip.

Consider the broken and unbroken fiber configuration shown in Fig. 2. The original crack size is b and unbroken fibers between the right crack tip and the broken fiber in the right end are denoted as $i_1, i_2, i_3 \ldots i_t (b < i_1 < i_2 \ldots i_t < b')$. b' denotes the broken fiber in the right end. The stress concentration factors in these fibers are denoted as $C_{i\alpha} (\alpha = 1,2,\ldots t)$.

We define the coinfluence coefficients L^*_{nm} representing the displacement of the nth fiber at $\xi = +0$ when the unit force dipole is applied on the mth fiber at the crack plane. It is obvious that the matrix $[\tilde{\tilde{L}}^*]$ is the inverse of the matrix $[\tilde{\tilde{L}}]$,

$$[\tilde{\tilde{L}}] \cdot [\tilde{\tilde{L}}^*] = [\tilde{\tilde{L}}^*] \cdot [\tilde{\tilde{L}}] = [\tilde{\tilde{E}}] \tag{23}$$

where the matrix $[\tilde{\tilde{L}}^*]$ is composed of the components L^*_{nm} and the matrix $[\tilde{\tilde{L}}]$ is composed of the components L_{nm}. The matrix $[\tilde{\tilde{E}}]$ represents the $(b' \times b')$ unit matrix.

L_{nm} is given as

$$L_{nm} = (4/\pi) \cdot \{4(n-m)^2 - 1\}^{-1} \tag{24}$$

Using the coinfluence coefficient L^*_{nm} and the half-crack opening displacements $U_n^{b'} (n = 1,2,\ldots b')$, the stress concentration factor $C_{i\alpha}$ in the $i\alpha$th fiber $(\alpha = 1,2,\ldots t)$ is obtained as

$$\begin{bmatrix} C_{i1} \\ C_{i2} \\ \vdots \\ C_{it} \end{bmatrix} = - \begin{bmatrix} L^*_{i1,i1} & L^*_{i1,i2} & \cdots & L^*_{i1,it} \\ L^*_{i2,i1} & L^*_{i2,i2} & \cdots & L^*_{i2,it} \\ \vdots & \vdots & & \vdots \\ L^*_{it,i1} & L^*_{it,i2} & \cdots & L^*_{it,it} \end{bmatrix}^{-1} \cdot \begin{bmatrix} U_{i1}^{b'} \\ U_{i2}^{b'} \\ \vdots \\ U_{it}^{b'} \end{bmatrix} \tag{25}$$

L^*_{nm} may be evaluated by calculating the inverse matrix of the influence function matrix $[\tilde{\tilde{L}}]$. However, it is much simpler to calculate L^*_{nm} by the formula[15]

$$L^*_{nm} = -\frac{2}{\pi}\left\{ \sum_{k=n+1-m}^{b'+1-m} \frac{1}{2k-1} + \sum_{s=1}^{\infty} \left[K_{m,b'}^{b'+s} \cdot \left(\sum_{k=s}^{s+b'-n} \frac{1}{2k-1} \right) - K_{b'+1-m,b'}^{b'+s} \cdot \left(\sum_{k=s+n}^{s+b'} \frac{1}{2k-1} \right) \right] \right\} \quad (n \geq m) \tag{26}$$

where

$$K_{a,b}^{c} = \frac{1}{2} \cdot \frac{\{2(b-a)+1\}!! \cdot \{2a-1\}!! \cdot \{2(c-b)-3\}!! \cdot \{2(c-1)\}!!}{\{2(b-a)\}!! \cdot \{2(a-1)\}!! \cdot \{2(c-b-1)\}!! \cdot \{2c-1\}!! \cdot (c-a)} \tag{27}$$

Table 1 Limiting value of $-L^*_{nm}$

	m			
n	1	2	3	4
1	1.00	0.50	0.37	0.31
2	0.50	1.25	0.68	0.53
3	0.37	0.68	1.38	0.80
4	0.31	0.53	0.80	1.48

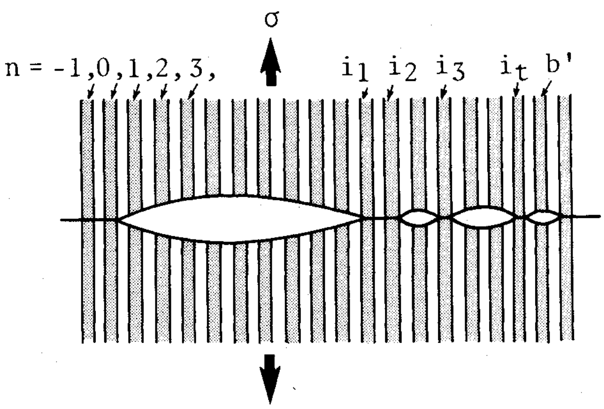

Fig. 2 Model of a multifilament crack after propagation.

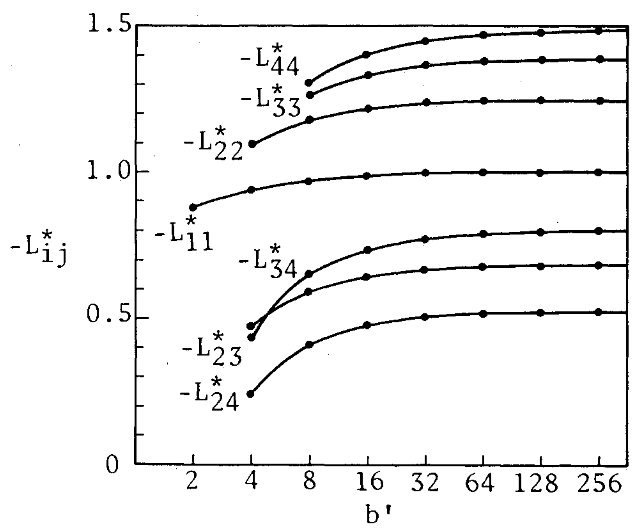

Fig. 3 $-L^*_{ij}$ vs crack size b' for $1 \leq i,j \leq 4$.

Figure 3 depicts $(-L^*_{nm})$ as the function of b' for $1 \le n, m \le 4$. Note that $(-L^*_{nm})$ increases as the crack size increases and then converges to a limiting value. As a special case L^*_{11} is obtained in closed form,

$$L^*_{11} = -\frac{\pi}{2} \cdot \frac{(2b'-1)!!}{(2b'-2)!!} \cdot \frac{(2b'-1)!!}{2b'!!} \tag{28a}$$

From Eq. (28a) convergence of L^*_{11} is proved with the aid of Sterling's formula,

$$\lim_{b' \to \infty} -L^*_{11} = \lim_{b' \to \infty} \frac{\pi}{2} \frac{(2b'-1)!!}{(2b'-2)!!} \cdot \frac{(2b'-1)!!}{(2b')!!} = 1 \tag{28b}$$

The convergent speed of $L^*_{nm}(1 \le n, m \le 4)$ in Fig. 3 is observed to be rather rapid. Thus, in the following section we replace L^*_{nm} by its limiting value, assuming our crack size is sufficiently large. Table 1 represents the limiting value of L^*_{nm} for every configuration of broken and unbroken fibers among the first four neighboring fibers at the crack tip.

The fiber stress concentration factors are calculated from Eq. (25), inserting Eq. (21) for $U^{b'}_{i\alpha}$. Note that the fiber stress concentration factors $C_{i\alpha}(\alpha = 1,2,\ldots t)$ are also given as a product of K^I_b and $C'_{i\alpha}$, which is independent of crack size b. Table 2 shows $C'_{i\alpha}(\alpha = 1,2,3,4)$ for every possible configuration of failed and unfailed fibers among the first four neighboring fibers at the crack tip.

Propagation of Crack

We shall now study the probability distribution for the propagation of a crack with b broken fibers under a uniform applied stress x at infinity. We define the crack propagation of n steps as an event that n successive fibers ahead of the original crack tip have totally failed. n is not the total number of broken fibers ahead of the original crack tip, but the number of successive broken fibers ahead of the original crack tip. For instance, example A shows that the first, third, and fourth fibers ahead of the crack tip have failed. This illustrates one-step crack propagation. Example B also illustrates one-step crack propagation. C and D are examples of two-step crack propagation. The cumulative distribution function for n steps crack propagation is denoted as $G_n(x)$. In the following, we obtain $G_n(x)$ from the cumulative strength distribution function $F(x)$ for each fiber. For simplicity, we assume $F(x)$ is given by the two-parameter Weibull function

$$F(x) = 1 - \exp[-(x/x_0)^\rho] \tag{29}$$

where x_0 is the scale parameter and ρ the shape parameter.

Our interest is in calculating $G_n(x)$ in a precise way, taking into account the possible sequences of steps up to the failure of the first n fibers neighboring the crack tip. However, it is almost impossible to count all of the step sequences. Here, we consider every possible failure sequence only within the first four fibers ahead of the crack tip. This assumption may be replaced by the weak one in which failure sequences in more than four fibers ahead of the crack tip are taken into account. However, we will see that consideration of the first four fibers gives a sufficiently good approximation. Equation (30) describes $G_1(x)$ and $G_3(x)$ using the method of failure sequences.[3] The superscript i in $G^i_1(x)$ and $G^i_3(x)$ indicates that every failure step in the first i fibers ahead of the crack tip is taken into consideration.

$$\begin{aligned} G^4_1(x) = 1 - \{ & (1-Q_{04})(1-Q_{03})(1-Q_{02})(1-Q_{01}) \\ & + (1-Q_{13})(1-Q_{12})(1-Q_{11})Q_{04} \\ & + (1-Q_{24})(1-Q_{22})(1-Q_{21})Q_{03} \\ & + (1-Q_{32})(1-Q_{31})\{Q_{04}Q_{13} + Q_{03}Q_{44} - Q_{04}Q_{03}\} \\ & + (1-Q_{44})(1-Q_{43})(1-Q_{41})Q_{02} \\ & + (1-Q_{53})(1-Q_{51})\{Q_{04}Q_{12} + Q_{02}Q_{44} - Q_{04}Q_{02}\} \\ & + (1-Q_{64})(1-Q_{61})\{Q_{03}Q_{22} + Q_{02}Q_{43} - Q_{03}Q_{02}\} \\ & + (1-Q_{71})\{Q_{04}(Q_{13}-Q_{03})(Q_{32}-Q_{02}) \\ & + Q_{04}(Q_{12}-Q_{02})(Q_{53}-Q_{13}) + Q_{03}(Q_{24}-Q_{04})(Q_{32}-Q_{02}) \\ & + Q_{03}(Q_{22}-Q_{02})(Q_{64}-Q_{24}) + Q_{02}(Q_{44}-Q_{04})(Q_{53}-Q_{03}) \\ & + Q_{02}(Q_{43}-Q_{03})(Q_{64}-Q_{44}) + Q_{04}Q_{02}Q_{32} \\ & + Q_{03}Q_{02}(Q_{64}-Q_{04}) + Q_{02}Q_{04}(Q_{53}-Q_{03})\}\} \end{aligned} \tag{30}$$

where $Q_{ij} = F(C_{b+j}X)$ for the failure mode S_i in Table 2. For the purpose of comparison with $G^4_1(x)$, we have also calculated $G^1_1(x)$, $G^2_1(x)$, and $G^3_1(x)$ as follows:

$$G^1_1(x) = Q_{01} \tag{31}$$

$$G^2_1(x) = Q_{01} + (Q_{41} - Q_{01})Q_{02} \tag{32}$$

$$\begin{aligned} G^3_1(x) = Q_{01} & + (Q_{21} - Q_{01})(Q_{03} + Q_{02} - Q_{03}Q_{02}) \\ & + (Q_{41} - Q_{21})(Q_{02} + Q_{03}Q_{22} - Q_{03}Q_{02}) \\ & + (Q_{61} - Q_{41})(Q_{03}Q_{22} + Q_{02}Q_{43} - Q_{03}Q_{02}) \end{aligned} \tag{33}$$

$G_n(x)$ for the higher-step crack propagation is also derived in the same manner,

$$G^1_3(x) = Q^3_{01} \tag{34}$$

Table 2 Stress concentration factors

Configuration of failed and unfailed[a] fibers (s=1 2 3 4)		C_{b+1}/K^I_b	C_{b+2}/K^I_b	C_{b+3}/K^I_b	C_{b+4}/K^I_b
S_0	(1 1 1 1)	1.00	0.500	0.375	0.313
S_1	(1 1 1 0)	1.02	0.532	0.478	—
S_2	(1 1 0 1)	1.04	0.637	—	0.442
S_3	(1 1 0 0)	1.08	0.731	—	—
S_4	(1 0 1 1)	1.21	—	0.547	0.352
S_5	(1 0 1 0)	1.24	—	0.676	—
S_6	(1 0 0 1)	1.36	—	—	0.579
S_7	(1 0 0 0)	1.48	—	—	—

[a]The numbers 1 and 0 in parentheses indicate unfailed and failed fiber, respectively.

Table 3 Example of n steps crack propagation

Example	Starting condition $m=1...b$	$b+1$	$b+2$	$b+3$	$b+4$	Ending condition $n=1...b$	$b+1$	$b+2$	$b+3$	$b+4$
A	0...0	1	1	1	1	0...0	0	1	0	0
B	0...0	1	1	1	1	0...0	0	1	1	1
C	0...0	1	1	1	1	0...0	0	0	1	0
D	0...0	1	1	1	1	0...0	0	0	1	1

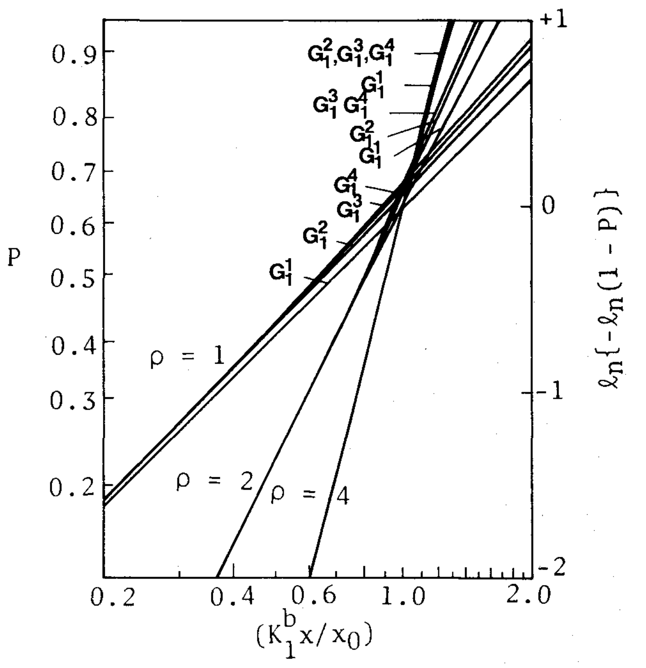

Fig. 4 Cumulative distribution function for one-step crack propagation $G_l(x)$ for Weibull shape parameter $\rho=1$, 2, and 4.

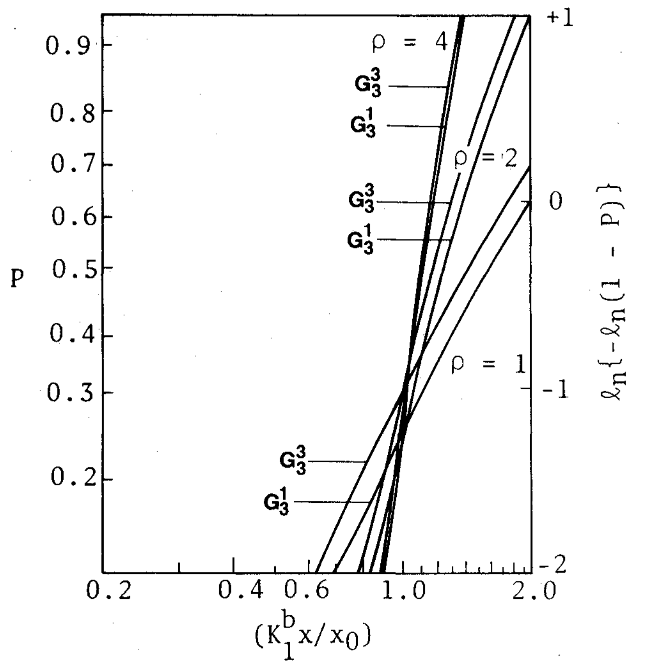

Fig. 5 Cumulative distribution function for three-step crack propagation $G_3(x)$ for $\rho=1$, 2, and 4.

$$G_3^3(x)=Q_{03}(Q_{22}-Q_{02})(Q_{6l}-Q_{0l})+Q_{03}(Q_{2l}-Q_{0l})$$
$$\times(Q_{4l}-Q_{22})+Q_{02}(Q_{4l}-Q_{0l})(Q_{0l}-Q_{03})$$
$$+Q_{02}(Q_{43}-Q_{03})(Q_{6l}-Q_{5l})+Q_{0l}(Q_{4l}-Q_{02})$$
$$\times(Q_{02}-Q_{03})+Q_{0l}(Q_{0l}-Q_{02})^2+Q_{03}Q_{02}Q_{6l}$$
$$+Q_{0l}Q_{02}(Q_{0l}-Q_{03})+Q_{0l}Q_{03}(Q_{4l}-Q_{02}) \tag{35}$$

The resulting graphs are displayed on Figs. 4 and 5 for the shape parameter of fiber strength distribution $\rho=1$, 2, and 4. We make the following observations:

1) The values $\rho=1$ and 2 correspond respectively to coefficients of variation of 1.0 and 0.52 in fiber strength that are quite large. Observe in Fig. 4 that the plots of G_l^4 are 7% higher than the plots of G_l^l in the probability region around the mean value. Also observe that the plots of $G_l^i(x)$ differ an insignificant amount for $i=3$, 4. Thus, there seems to be convergence in $G_l^i(x)$ as i increases.

2) The large value of ρ corresponds to a small coefficient of variation in fiber strength. For $\rho=4$, the coefficient of variation in fiber strength is 0.28. In this case, the possibility of failure sequences other than the single failure of the first intact fiber at the crack tip (see example B in Table 3) is rather small. For $\rho\geq4$, $G_l^{(i)}$ $(i=1,2,3,4)$ are virtually identical.

3) In Fig. 5, the cumulative distribution function for the three-step crack propagation $G_3(x)$ is shown. The difference between $G_3^l(x)$ and $G_3^3(x)$ is more striking than the difference between $G_l^l(x)$ and $G_l^3(x)$. The reason is that the number of additive terms contributing significantly to the total numerical probability increases rapidly with the number of steps in the crack propagation.

The numerical results indicate that if the coefficient of variation of fiber strength is large ($\rho<4$), we need to consider the failure step ahead of the first intact fiber precisely. The calculation of $G_n(x)$ for an arbitrary step n is presently not possible. However, the following observations of the lower tail analysis give some insight into the behavior of $G_n(x)$ as n increases.

An expression for the lower tail of the Weibull distribution is obtained by expanding $F(x)$ in a Taylor series about $x=0$, giving

$$F(x)=(x/x_0)^\rho+O[x/x_0]^\rho, \qquad x\geq0 \tag{36}$$

where $O[z]$ is defined by $O[z]/z\to0$ as $z\to0$. An expression for the lower tail of $G_3^3(x)$ is then obtained by substituting Eq. (36) into Eq. (35),

$$G_3^3(x)=1.27w^{3\rho}+O[w^{3\rho}] \tag{37}$$

where $w=K_b^l x/x_0$.

Then consider the cumulative distribution function for the probability of $3n$ steps crack propagation $G_{3n}(x)$. We assume $b\gg3n\gg1$. The $3n$ steps crack propagation includes one group of failure sequences, each of which consists of n times the three-step crack propagation. Thus, $G_{3n}(x)$ is evaluated as

$$G_{3n}(x)\geq[G_{3n}^3(x)]^n \tag{38}$$

Inserting Eq. (37) into Eq. (38) we obtain

$$G_{3n}(x) \geq 1.27^n w^{3n\rho} + O[w^{3n\rho}] \tag{39}$$

As the number of steps $3n$ increases, the lower tail of G_{3n} becomes large compared to the lower tail of G^l_{3n} [Eq. (40)], which represents the $3n$-step crack propagation by the successive failures of the fiber in contact with the crack tip.

$$G^l_{3n}(x) = w^{3n\rho} + O[w^{3n\rho}] \tag{40}$$

The above analysis supports the observation in Figs. 4 and 5 that the probability for the failure sequences other than the successive failures of the fiber in contact with the crack tip increases rapidly as the number of steps in the crack propagation increases. A more detailed investigation of these aspects will be considered in a future publication.

Conclusions

The failure process of unidirectionally reinforced fiber composites with a transverse slit notch has been studied theoretically, based upon the solution of two-dimensional crack problems by the shear-lag method. The main findings are:

1) For a large crack, the fiber stress and its displacement around the crack tip are given as the product of the fiber stress concentration factor K^l_b at the crack tip and a local factor independent of the crack size.

2) The stress redistribution after the failure of some fibers ahead of the first intact fiber is also given as the product of K^l_b and a local factor independent of crack size. Thus, the probability for the crack propagation is expressed as the function of $K^l_b x$, where x is the applied stress.

3) The failure sequence of unidirectionally reinforced fiber composites is affected by the variability in fiber strength. When the variation in fiber strength is large, say the Weibull shape parameter ρ of the cumulative fiber strength distribution is less than four, then the possibility of failure starting from the fibers apart from the crack tips is not negligible.

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